Minimal fibrations and the organizing theorem of simplicial homotopy theory

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Abstract

Quillen showed that simplicial sets form a model category (with appropriate choices of three classes of morphisms), which organized the homotopy theory of simplicial sets. His proof is very difficult and uses even the classification theory of principal bundles. Thus, Goerss-Jardine appealed to topological methods for the verification. In this paper we give a new proof of this organizing theorem of simplicial homotopy theory which is elementary in the sense that it does not use the classifying theory of principal bundles or appeal to topological methods.

1 Introduction

The homotopy theory of simplicial sets was developed mostly by Kan in the 1950's. His work made homotopy theory independent of general topology. In the 1960's Quillen [13] organized the homotopy theory of simplicial sets in the framework of model categories which is a modern foundation of homotopy theory. The category of simplicial sets is not only an example of a model category but also an indispensable ingredient of model category theory (cf. [4-6]). Thus simplicial sets and their model structure have become essential in various field of mathematics as model categories have been important in abstract homotopy theory and its applications [2, 7, 10-12].

The organizing theorem of simplicial homotopy theory asserts that simplicial sets form a model category. By definition a model category is just an ordinary category with three specified classes of morphisms, called fibrations, cofibrations and weak equivalences, which satisfy several axioms. Thus the proof of the organizing theorem is nothing but verification of the model axioms for the category of simplicial sets and it is, as Goerss and Jardine [4, p. 2] say, "still one of the more difficult proofs of abstract homotopy theory" (see also [6, p. 73]). Quillen's method [13] of verification is purely simplicial. But he was "unable to find a really elementary proof"; it uses even the classification theory of principal bundles due to Barratt-Guggenheim-Moore [1] ([13, Introduction of Chapter II]). Thus Goerss and Jardine [4, Chapter I] appeal to topological methods to show the organizing theorem of simplicial homotopy theory (especially note the definition of a weak equivalence [4, p. 60] and the

result on the realization of a Kan fibration [4, pp. 54-59]). But the principle of simplicial homotopy theory is to develope homotopy theory excluding topological methods, hence the proof of the organizing theorem should be purely simplicial but not topological.

Main objective of this paper is to give a proof of the following organizing theorem of simplicial homotopy theory which is elementary in the sense that it does not use the classifying theory of principal bundles or appeal to topological methods.

Let $\mathbf{S}$ denote the category of simplicial sets. If Z is a Kan complex, the homotopy set $[X, Z]$ is defined as the set of homotopy classes of maps from X to Z (see Section 2).

Theorem 1.1. *Define a map $f : X \longrightarrow Y$ in $\mathbf{S}$ to be*
(1) a fibration if the map f is a Kan fibration,
(2) a weak equivalence if the map f induces a bijection $f^{\#} : [Y, Z] \longrightarrow [X, Z]$ for any Kan complex Z, and
(3) a cofibration if the map f has the left lifting property with respect to each map which is both a fibration and a weak equivalence.
Then with these choices $\mathbf{S}$ is a model category.

In fact we show that the category $\mathbf{S}$ with the specified classes of morphisms satisfies stronger model axioms, which are adopted in [5] (c.f. Section 2). In other words the category $\mathbf{S}$ is complete and cocomplete, and the factorizations described in the so-called factorization axiom are functorially constructed.

We adopt the definition of a weak equivalence different from those of Quillen [13, 3.14 in Chapter II] and Goerss-Jardine [4, p. 60], which is a key to simplifying the proof of the organizing theorem. We also check that our model structure coincides with theirs.

Proposition 1.2. *The model structure of $\mathbf{S}$ in Theorem 1.1 coincides with those of Quillen and Goerss-Jardine.*

The difficulty of checking the model axioms for the category $\mathbf{S}$ is having to deal with general simplicial sets which need not to be Kan complexes, which is different from the case of the category of topological spaces. Thus we make effective use of the powerful theory of minimal fibrations whose topological analogue does not exist (see the arguments used in the proofs of Lemmas 2.4 and 3.3).

The proof of Theorem 1.1 is given in Section 2; a purely simplicial proof of Lemma 2.4 is crucial. In section 3, we investigate cofibrations and weak equivalences in the category $\mathbf{S}$. The investigation of cofibrations implies Proposition 1.2. An alternative proof of Lemma 2.4, which is the heart of the proof of Theorem 1.1, is given in Section 4. One of the advantage of this proof is that it does not need any knowledge of higher homotopy groups and homotopy exact sequences.

Refer to [4] for the fundamental notions and results of simplicial homotopy theory and [3] for the basic facts and the terminology of model category theory. (See also [4] and [5] for more details on model categories.) The recently published book [9] by May and Ponte is also relevant to our subject. It contains another simplified proof of the organizing theorem of simplicial homotopy theory, which makes a little use of topology.
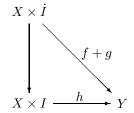
2   Proof of Theorem 1.1

We begin by recalling the standard notations and fundamental notions of the theory of simplicial sets.

Let Δ^p denote the *standard p-simplex* and Δ^p_i its *i-th face*. Let $\dot{\Delta}^p$ and Λ^p_k denote the *boundary* and the *k-th horn* of Δ^p respectively. Thus $\dot{\Delta}^p$ is just the subcomplex of Δ^p generated by Δ^p_i $(0 \leq i \leq p)$ and Λ^p_k is just the subcomplex of Δ^p generated by Δ^p_i $(0 \leq$

$i \leq p, i \neq k$). The standard 1-simplex Δ^1 and its boundary $\dot{\Delta}^1$ are also denoted by I and $\dot{I}$ respectively.

A simplicial map $f : X \longrightarrow Y$ is called a *Kan fibration* iff f has the right lifting property with respect to standard inclusions $\Lambda_k^p \longrightarrow \Delta^p$ ($p > 0$, $0 \leq k \leq p$). A simplicial set X is called a *Kan complex* iff the canonical map from X to a terminal object $*$ is a Kan fibration.

Let f, $g : X \longrightarrow Y$ be simplicial maps. We say f is *homotopic* to g, written $f \simeq g$, if there exists a commutative diagram

$$
\begin{array}{ccc}
X \times \dot{I} & & \\
\Big\downarrow & \searrow^{f+g} & \\
X \times I & \xrightarrow{\ h\ } & Y
\end{array}
$$

where the vertical arrow is the product of 1_X and the standard inclusion. The map h is called a *homotopy*. If Y is a Kan complex, the hotomopy relation $\simeq$ is an equivalence relation ([4, Lemma 6.1 in Chapter I]). Thus the *homotopy set* $[X, Y]$ is defined as the quotient set $\mathrm{Hom}(X,Y)/\simeq$.

We make full use of the Gabriel-Zisman theory of anodyne extensions ([4, Section 4 in Chapter I]) which engulfs old combinatorial arguments.
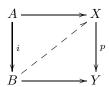
Let us verify the following model axioms which are stronger than Quillen's as stated in Section 1:

MC1: **S** is closed under all small limits and colimits.

MC2: If f and g are maps in **S** such that gf is defined and if two of the three maps f, g, gf are weak equivalences, then so is the third.

MC3: If f is a retract of g and g is a weak equivalence, fibration, or cofibration, then so is f.

MC4: Given a commutative solid arrow diagram

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
\Big\downarrow{\scriptstyle i} & \nearrow & \Big\downarrow{\scriptstyle p} \\
B & \longrightarrow & Y
\end{array}
$$

the dotted arrow exists, making the diagram commute, if either

(i) i is a cofibration and p is an acyclic fibration (i.e., a fibration that is also a weak equivalence), or

(ii) i is an acyclic cofibration (i.e., a cofibration that is also a weak equivalence) and p is a fibration.
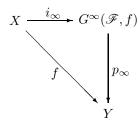
MC5: Every map f has two functorial factorizations:

(i) $f = qj$, where j is a cofibration and q is an acyclic fibration, and

(ii) $f = pi$, where i is an acyclic cofibration and p is a fibration.

Since $\mathbf{S} = Set^{\Delta^{op}}$, limits and colimits are objectwise constructed. Thus **MC1** is satisfied. **MC2** is obvious. **MC3** is not difficult (cf. [3, 2.7 and 8.10]).

To prove the factorization axiom **MC5**(ii), note that $f : X \longrightarrow Y$ is a fibration iff f has the right lifting property with respect to standard inclusions $\Lambda_k^p \longrightarrow \Delta^p$ ($p > 0$, $0 \leq k \leq p$). Since Λ_k^p is sequentially small, we can apply the small object argument ([3, p. 104]) to

$\mathscr{F} = \{\Lambda_k^p \longrightarrow \Delta^p \ (p > 0, \ 0 \leq k \leq p)\}$. Then we obtain a factorization

$$
\begin{array}{ccc}
X & \xrightarrow{\ i_\infty\ } & G^\infty(\mathscr{F}, f) \\
& \searrow{\scriptstyle f} & \downarrow{\scriptstyle p_\infty} \\
& & Y
\end{array}
$$

such that p_∞ is a fibration and $G^\infty(\mathscr{F}, f)$ is the direct limit of a sequence of the form

$$
X = G^0(\mathscr{F}, f) \xhookrightarrow{\ i_1\ } G^1(\mathscr{F}, f) \xhookrightarrow{\ i_2\ } \cdots \xhookrightarrow{\ i_n\ } G^n(\mathscr{F}, f) \lhook\joinrel\longrightarrow \cdots
$$

where each i_n fits into a pushout diagram of the form

$$
\begin{array}{ccc}
\coprod_{\lambda \in \Lambda_n} \Lambda_{k_\lambda}^{p_\lambda} & \lhook\joinrel\longrightarrow & \coprod_{\lambda \in \Lambda_n} \Delta^{p_\lambda} \\
\downarrow & & \downarrow \\
G^{n-1}(\mathscr{F}, f) & \xhookrightarrow{\ i_n\ } & G^n(\mathscr{F}, f)
\end{array} \ .
$$

Since the class of anodyne extensions is saturated by definition, i_∞ is an anodyne extension. Thus we see that this factorization is the desired one by the following lemma.

Lemma 2.1. *Let $i : A \longrightarrow B$ be an anodyne extension. Then i is an acyclic cofibration.*

Proof. Since $i : A \longrightarrow B$ is an anodyne extension, the inclusion

$$
A \times I \cup B \times \dot{I} \lhook\joinrel\longrightarrow B \times I
$$

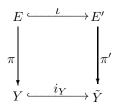is also an anodyne extension by [4, Corollary 4.6 in Chapter I]. Therefore

$$
i^\sharp : [B, Z] \longrightarrow [A, Z]
$$

is bijective for any Kan complex Z (cf. [4, Corollary 4.3 in Chapter I]), which shows that i is a weak equivalence. Since an anodyne extension is a cofibration by [4, Corollary 4.3 in Chapter I], i is an acyclic cofibration. $\qquad\square$

Let us make preparations to prove the factorization axiom **MC5**(i). A simplicial set X is said to be *fibrant* if the map from X to a terminal object $*$ is a fibration. Note that X is fibrant if and only if X is a Kan complex. For each simplicial set X, we can apply the (functorial) "acyclic cofibration-fibration" factorization axiom **MC5**(ii) to the map from X to $*$ and obtain an acyclic cofibration $i_X : X \hookrightarrow \tilde{X}$ with $\tilde{X}$ fibrant. We call $\tilde{X}$ the *fibrant approximation* to X. A simplicial map $\pi : E \longrightarrow Y$ is called an *F-bundle* if the pull-back $f^{-1}E$ is isomorphic to $\Delta^n \times F$ over Δ^n for any n and any map $f : \Delta^n \longrightarrow Y$.

Lemma 2.2. *Let F and Y be simplicial sets.*
(1) Let $\pi : E \longrightarrow \Lambda_k^p$ be an F-bundle. Then E is a trivial F-bundle.
(2) Let $\pi : E \longrightarrow Y$ be an F-bundle. Then there exists a cartesian square

$$
\begin{array}{ccc}
E & \xhookrightarrow{\ \iota\ } & E' \\
\pi \downarrow & & \downarrow \pi' \\
Y & \xhookrightarrow{\ i_Y\ } & \tilde{Y}
\end{array}
$$

such that $\pi' : E' \longrightarrow \tilde{Y}$ *is an F-bundle and ι is an acyclic cofibration.*

Proof. (1) Since E is trivial over each face Δ_i^p $(i \neq k)$, we can take trivializations $E|_{\Delta_i^p} \xrightarrow[\cong]{\varphi_i}$ $\Delta_i^p \times F$. Consider the simplicial group $Aut(F)$ whose p-simplices are the commutative diagrams of the form

$$\Delta^p \times F \xrightarrow[\cong]{\varphi} \Delta^p \times F$$
$$\searrow \qquad \swarrow$$
$$\Delta^p$$

and note that it is a Kan complex (cf. [4, p. 12]). Then we can modify $\{\varphi_i\}$ so that $\cup \varphi_i$ gives a trivialization of E.

(2) By (1), we can easily extend $\pi : E \longrightarrow Y$ to an F-bundle over $\tilde{Y}$, which is denoted by π' : $E' \longrightarrow \tilde{Y}$. Then E' is the direct limit of a sequence of the form

$$E = E(0) \xrightarrow{\iota_1} E(1) \xrightarrow{\iota_2} \cdots \xrightarrow{\iota_n} E(n) \longrightarrow \cdots$$

where each ι_n fits into a pushout diagram of the form

$$
\begin{array}{ccc}
\coprod_{\lambda \in \Lambda_n} (\Lambda_{k_\lambda}^{p_\lambda} \times F) & \longrightarrow & \coprod_{\lambda \in \Lambda_n} (\Delta^{p_\lambda} \times F) \\
\downarrow & & \downarrow \\
E(n-1) & \xrightarrow{\iota_n} & E(n)
\end{array}
$$

Note that $\Lambda_{k_\lambda}^{p_\lambda} \times F \hookrightarrow \Delta^{p_\lambda} \times F$ is an anodyne extension by [4, Corollary 4.6 in Chapter I]. Since the class of anodyne extensions is saturated by definition, ι is an anodyne extension, and hence an acyclic cofibration by Lemma 2.1. $\square$

Under our definition of a weak equivalence, we show

Lemma 2.3. *Let $f : X \longrightarrow Y$ be a weak equivalence between Kan complexes. Then f is a homotopy equivalence.*

Proof. Since f is a weak equivalence, $f^\sharp : [Y, Z] \longrightarrow [X, Z]$ is bijective for any Kan complex Z. Set $Z = X$. Then there exists a unique homotopy class $[g]$ such that $f^\sharp([g]) = [1_X]$ and hence $g \circ f \simeq 1_X$. Thus $g^\sharp$ is a right inverse to $f^\sharp : [Y, Z] \longrightarrow [X, Z]$ for any Kan complex Z. Since $f^\sharp$ is bijective, $g^\sharp \circ f^\sharp = 1$ also holds on $[Y, Z]$. By setting $Z = Y$ and evaluating this identity at $[1_Y]$, we have $f \circ g \simeq 1_Y$. $\square$
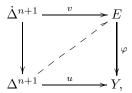
We use Lemmas 2.2 and 2.3 to prove the following lemma which is necessary to apply the small object argument and obtain the desired factorization.

Lemma 2.4. *A map $f : X \longrightarrow Y$ is an acyclic fibration if and only if f has the right lifting property with respect to standard inclusions $\dot{\Delta}^n \longrightarrow \Delta^n$ $(n \geq 0)$.*

Proof. ($\Longleftarrow$) Suppose that f has the right lifting property with respect to $\{\dot{\Delta}^n \longrightarrow \Delta^n \ (n \geq 0)\}$. The standard inclusion $\Lambda_k^n \longrightarrow \Delta^n$ is the composite
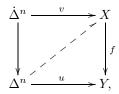
$$\Lambda_k^n \longrightarrow \dot{\Delta}^n \longrightarrow \Delta^n$$

and $\Lambda_k^n \longrightarrow \dot{\Delta}^n$ is a pushout of $\dot{\Delta}^{n-1} \longrightarrow \Delta^{n-1}$. Hence it follows that f is a fibration (cf. [4, Lemma 4.1 in Chapter I]). Thus we take a minimal subfibration $\varphi : E \longrightarrow Y$ of f. Since E is a vertical deformation retract of X, φ also has the right lifting property with respect to $\{\dot{\Delta}^n \longrightarrow \Delta^n \ (n \geq 0)\}$. Suppose that the fiber M of φ has nontrivial π_n for some n and some base point m. Then we can take a nontrivial element $[h] \in \pi_n(M, m)$ and consider a lifting problem
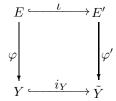
$$\begin{array}{ccc} \dot{\Delta}^{n+1} & \xrightarrow{\ v\ } & E \\ \downarrow & \nearrow & \downarrow{\scriptstyle \varphi} \\ \Delta^{n+1} & \xrightarrow{\ u\ } & Y, \end{array}$$

where u is the constant map to $\varphi(m)$ and v is the sum of h on the $(n+1)$-th face and the constant map to m on the $(n+1)$-th horn. It is clearly unsolvable, which is a contradiction. Thus M is a minimal complex whose homotopy groups are trivial. By the construction and the uniqueness of a minimal subcomplex [4, Proposition 10.3 and Lemma 10.4 in Chapter I], M is a terminal object, hence $\varphi : E \longrightarrow Y$ is an isomorphism. Therefore $f : X \longrightarrow Y$ is a homotopy equivalence and hence a weak equivalence.

($\Longrightarrow$) Suppose that f is an acyclic fibration. Then, for a given commutative solid arrow diagram

$$\begin{array}{ccc} \dot{\Delta}^n & \xrightarrow{\ v\ } & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \Delta^n & \xrightarrow{\ u\ } & Y, \end{array}$$

we must construct the dotted arrow making the diagram commute. We take a minimal subfibration $\varphi : E \longrightarrow Y$ of $f : X \longrightarrow Y$ and write the inclusion i. We fix a vertical retraction $r : X \longrightarrow E$ and a vertical homotopy $R : X \times I \longrightarrow X$ from 1_X to ir such that the composite $E \times I \xrightarrow{i \times 1} X \times I \xrightarrow{R} X$ is the constant homotopy of i.

First let us show that φ is an isomorphism. Note that φ is a fiber bundle whose fiber M is a minimal complex ([4, p. 47 and Collary 10.8 in Chapter I]). Thus we have an extension of an M-bundle φ

$$\begin{array}{ccc} E & \xhookrightarrow{\ \iota\ } & E' \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \varphi'} \\ Y & \xhookrightarrow{\ i_Y\ } & \tilde{Y} \end{array}$$

by Lemma 2.2. Since φ' is a fiber bundle with Kan complex fiber, φ' is a fibration. Thus both of $\tilde{Y}$ and E' are Kan complexes and φ' is a homotopy equivalence by Lamma 2.3. Hence M has trivial homotopy groups by the homotopy exact sequence ([4, Lemma 7.3 in Chapter I]). This implies that M is a terminal object, which shows that φ is an isomorphism.

Now consider the following lifting problem

$$\begin{array}{ccc} \dot{\Delta}^n \times I \cup \Delta^n \times (1) & \xrightarrow{\ V\ } & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ \Delta^n \times I & \xrightarrow{\ U\ } & Y \end{array}$$

where U is the composite

$$\Delta^n \times I \xrightarrow{\;proj\;} \Delta^n \xrightarrow{\;\;u\;\;} Y$$
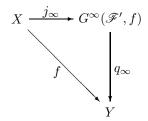
and V is the sum of

$$\dot\Delta^n \times I \xrightarrow{\;v \times 1\;} X \times I \xrightarrow{\;\;R\;\;} X$$

and

$$\Delta^n \times (1) \xrightarrow{\;can.lifting\;} E \overset{i}{\hookrightarrow} X \;.$$

(Since φ is an isomorphism, the canonical lifting of u is determined.) Then we obtain the dotted arrow making the diagram commute since the left vertical arrow is an anodyne extension. Its restriction to $\Delta^n \times (0)$ is a solution of the original lifting problem. $\qquad\square$

Since $\dot\Delta^n$ is sequentially small, we can apply the small object argument ([3, p. 104]) to $\mathscr{F}' = \{\dot\Delta^n \longrightarrow \Delta^n \ (n \geq 0)\}$. Then we obtain a factorization
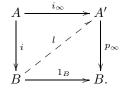
$$\begin{array}{ccc}
X & \xrightarrow{\;j_\infty\;} & G^\infty(\mathscr{F}', f) \\[2pt]
& \searrow{\scriptstyle f} & \big\downarrow{\scriptstyle q_\infty} \\[4pt]
& & Y
\end{array}\qquad .$$

such that q_∞ is an acyclic fibration. Note that $\dot\Delta^n \longrightarrow \Delta^n$ is a cofibration by Lemma 2.4, and observe that the class of monomorphisms which are cofibrations are saturated (cf. [4, Lemma 4.1 in Chapter I]). Then it is seen that j_∞ is a cofibration. Thus we have the desired factorization, which proves the axiom **MC5**(i).

MC4(i) is immediate from the definition of cofibrations. For **MC4**(ii), suppose that $i : A \longrightarrow B$ is an acyclic cofibration; we have to show that i has the left lifting property with respect to fibrations. Use the construction in the proof of **MC5**(ii) to factor $i : A \longrightarrow B$ as a composite

$$A \xrightarrow{\;i_\infty\;} A' \xrightarrow{\;p_\infty\;} B$$

and consider the following commutative solid arrrow diagram

$$\begin{array}{ccc}
A & \xrightarrow{\;i_\infty\;} & A' \\[2pt]
{\scriptstyle i}\big\downarrow & \nearrow{\scriptstyle l} & \big\downarrow{\scriptstyle p_\infty} \\[4pt]
B & \xrightarrow[\;1_B\;]{} & B.
\end{array}$$

Since p_∞ is an acyclic fibration by **MC2**, the dotted arrow l exists, making the diagram commue. Thus we see that i is a retract of i_∞. As was seen in the proof of **MC5**(ii), i_∞ is an anodyne extension, and hence so is i. Thus i has the left lifting property with respect to fibrations (cf. [4, Corollary 4.3 in Chapter I]).

Remark 2.1. In the proof of **MC4**(ii), we have shown that any acyclic cofibration is an anodyne extension. The converse is shown in Lemma 2.1. Therefore the class of acyclic cofibrations in the model category **S** is just the class of anodyne extensions.

3 Cofibrations and weak equivalences in S

Let us begin by identifying the cofibrations in the model category **S**. We denote by $\emptyset$ an initial object in the category **S**. A simplicial set X is said to be *cofibrant* if the map from $\emptyset$ to X is a cofibration.

Proposition 3.1. *Let $i : A \longrightarrow B$ be a map in **S**. Then the following are equivalent:*

(i) *i is a cofibration.*
    (ii) *i is a monomorphism.*
    (iii) *i is a degreewise injection.*

In particular, any simplicial set is cofibrant.

Proof. The equivalence of (ii) and (iii) is obvious. For the equivalence of (i) and (iii), the same arguments as in [13, 2.1 and 3.15 in Chapter II] apply. $\square$

*Proof of Proposition 1.*2. In a model category, the class of fibrations and the class of cofibrations determine the class of weak equivalences ([5, Proposition 7.2.7]). Thus Proposition 3.1 and the definition of fibrations imply that our model category structure on **S** coincides with those of Quillen and Goerss-Jardine. $\square$

Next, let us identify the weak equivalences in **S**. A map $f : X \longrightarrow Y$ between Kan complexes is called a weak homotopy equivalence if $f_\sharp : \pi_0(X) \longrightarrow \pi_0(Y)$ is a bijection and $f_\sharp : \pi_i(X, x) \longrightarrow \pi_i(Y, f(x))$ is an isomorphism for any 0-simplex x of X and any $i > 0$.

Proposition 3.2. *Let $f : X \longrightarrow Y$ be a map in **S**. Then the following are equivalent:*

(i) *f is a weak equivalence.*
    (ii) *$\tilde{f} : \tilde{X} \longrightarrow \tilde{Y}$ is a homotopy equivalence.*
    (iii) *$\tilde{f} : \tilde{X} \longrightarrow \tilde{Y}$ is a weak homotopy equivalence.*

By Lemma 2.3, it is enough to show the following lemma. It is a simplicial analogue of the Whitehead theorem and is proven in [8, §12]. But we give a simple proof using the model structure of the category **S**.

Lemma 3.3. *Let $f : X \longrightarrow Y$ be a map between Kan complexes. Then the following are equivalent:*

(i) *$f : X \longrightarrow Y$ is a homotopy equivalence.*
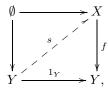(ii) *$f : X \longrightarrow Y$ is a weak homotopy equivalence.*

Proof. (i) $\Longrightarrow$ (ii) Obvious.
(ii) $\Longrightarrow$ (i) By the functorial factorization of **MC5**(ii) and Lemma 2.3, we can assume that $f : X \longrightarrow Y$ is a fibration. Take a minimal subfibration $\varphi : E \longrightarrow Y$ of f and a vertical deformation retraction $r : X \longrightarrow E$. Since the fiber M of φ is a minimal complex whose homotopy groups are trivial, M is a terminal object, hence $\varphi : E \longrightarrow Y$ is an isomorphism. Note that $f = \varphi r$ holds, and that r is a homotopy equivalence. Then it follows that f is a homotopy equivalence. $\square$

4 Alternative proof of Lemma 2.4

Proof. ($\Longleftarrow$) Suppose that f has the right lifting property with respect to $\{\dot{\Delta}^n \longrightarrow \Delta^n (n \geq 0)\}$. From the skeletal decomposition of a degreewise injection (the relative version of the decomposition in [4, p. 8]) and [4, Lemma 4.1 in Chapter I], f has the right lifting property with respect to the class of degreewise injections. From this, let us see that f is an acyclic fibration.

It is easily seen that f is a fibration. In order to see that f is a weak equivalence, take a solution s to a lifting problem

which gives rise to a retract diagram

$$Y \xrightarrow{s} X \xrightarrow{f} Y$$

in the overcategory $\mathbf{S}/Y$. Next, take a solution h to a lifting problem

which shows that Y is a deformation retract of X over Y. Thus f is a homotopy equivalence and hence a weak equivalence.

($\Longrightarrow$) Suppose that f is an acyclic fibration and $\varphi' : E' \longrightarrow \tilde{Y}$ is the M-bundle constructed in the original proof. First let us show that there exists a solution l to a lifting problem of the form
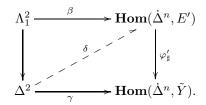

Since φ' is surjective, we can take a lifting l_0 of u. Since φ' is a homotopy equivalence by Lemma 2.3, we have $l_0 \mid_{\dot{\Delta}^n} \simeq v$. Thus we can take a 1-simplex m_0 of the function complex $\mathbf{Hom}(\dot{\Delta}^n, E')$ connecting $l_0 \mid_{\dot{\Delta}^n}$ to v. Note that $\varphi'_\sharp : \mathbf{Hom}(\dot{\Delta}^n, E') \longrightarrow \mathbf{Hom}(\dot{\Delta}^n, \tilde{Y})$ is a fibration which is a homotopy equivalence between Kan complexes (cf. [4, p. 21]), and that $a := \varphi'_\sharp(m_0)$ determines an element of $\pi_1(\mathbf{Hom}(\dot{\Delta}^n, \tilde{Y}), u \mid_{\dot{\Delta}^n})$. Thus we take a representative $\tilde{b}$ of $\pi_1(\varphi'_\sharp)^{-1}([a]^{-1}) \in \pi_1(\mathbf{Hom}(\dot{\Delta}^n, E'), v)$ and set $b = \varphi'_\sharp(\tilde{b})$. Then the map

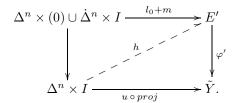$$(b, 0, a) : \dot{\Delta}^2 \longrightarrow \mathbf{Hom}(\dot{\Delta}^n, \tilde{Y})$$

has a filler $\gamma : \Delta^2 \longrightarrow \mathbf{Hom}(\dot{\Delta}^2, \tilde{Y})$, where $(b, 0, a)$ denotes the map whose restrictions to the 0-th, 1st, and 2nd faces are b, the constant map to $u \mid_{\dot{\Delta}^n}$, and a respectively. Let β denote

the sum of m_0 and $\tilde{b}$, and consider the lifting problem

$$
\begin{array}{ccc}
\Lambda_1^2 & \xrightarrow{\ \beta\ } & \mathbf{Hom}(\dot{\Delta}^n, E') \\
\downarrow & {\raise2pt\hbox{$\scriptstyle \delta$}} \nearrow & \downarrow{\scriptstyle \varphi'_\sharp} \\
\Delta^2 & \xrightarrow[\ \gamma\ ]{} & \mathbf{Hom}(\dot{\Delta}^n, \tilde{Y}).
\end{array}
$$

Since $\varphi'_\sharp$ is a fibration, there exists a solution δ. Then the restriction of δ to the first face gives a vertical homotopy m connecting $\ell_0 \mid_{\dot{\Delta}^n}$ to v.

Next, consider the lifting problem

$$
\begin{array}{ccc}
\Delta^n \times (0) \cup \dot{\Delta}^n \times I & \xrightarrow{\ l_0 + m\ } & E' \\
\downarrow & {\raise2pt\hbox{$\scriptstyle h$}} \nearrow & \downarrow{\scriptstyle \varphi'} \\
\Delta^n \times I & \xrightarrow[\ u \circ proj\ ]{} & \tilde{Y}.
\end{array}
$$

It has a solution h since the left vertical arrow is an anodyne extension. Set $l = h \mid_{\Delta^n \times (1)}$, which is a solution of the original problem.

We can show that f has the right lifting property with respect to $\{\dot{\Delta}^n \longrightarrow \Delta^n (n \geq 0)\}$ by a similar argument to that used in the original proof. $\qquad\square$

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